Chapter 1

Introductory Elements to Financial Mathematics

1.1. The object of traditional financial mathematics

The object of traditional financial mathematics is the formalization of the exchange between monetary amounts that are payable at different times and of the calculations related to the evaluation of the obligations of financial operations regarding a set of monetary movements.

The reasons for such movements vary and are connected to: personal or corporate reasons, patrimonial reasons (i.e. changes of assets or liabilities) or economic reasons (i.e. costs or revenues). These reasons can be related to initiatives regarding any kind of goods or services, but this branch of applied mathematics considers only the monetary counterpart for cash or assimilated values[1].

The evaluations are founded on equivalences between different amounts, paid at different times in certain or uncertain conditions. In the first part of this book we will cover financial mathematics in a deterministic context, assuming that the monetary income and outcome movements (to which we will refer as "payment" with no distinction) will happen and in the prefixed amount. We will not consider in

1 The reader familiar with book-keeping concepts and related rules knows that each monetary movement has a real counterpart of opposite movement: a payment at time *x* (negative financial amount) finds the counterpart in the opening of a credit or in the extinction of a debt. In the same way, a cashing (positive financial amount) corresponds to a negative patrimonial variation or an income for a received service. The position considered here, in financial mathematics, looks to the undertaken relations and the economic reasons for financial payments.

this context *decision theory in uncertain conditions*, which contains *actuarial mathematics* and more generally the *theory of random financial operations*[2].

We suppose that from now, unless otherwise specified, the deterministic hypotheses are valid, assuming then – in harmony with the rules of commonly accepted economic behavior – that:

a) the ownership of a capital (a monetary) amount is advantageous, and everyone will prefer to have it instead of not having it, whatever the amount is;

b) the temporary availability of someone else's capital or of your own capital is a favorable service and has a cost; it is then fair that whoever has this availability (useful for purchase of capital or consumer goods, for reserve funds, etc.) pays a price, proportional to the amount of capital and to the time element (the starting and closing dates of use, or only its time length).

The amount for the aforementioned price is called *interest.* The parameters used for its calculation are calculated using the rules of economic theory.

1.2. Financial supplies. Preference and indifference relations

1.2.1. *The subjective aspect of preferences*

Let us call *financial supply* a *dated amount*, that is, a prefixed amount to place at a given payment deadline. A supply can be formally represented as an ordinate couple (X,S) where S = *monetary amount* (transferred or accounted from one subject to another) and X = *time of payment.*

Referring to one of the contracting parties, S has an algebraic sign which refers to the cash flow; it is positive if it is an income and negative if it is an outcome, and the unit measure depends on the chosen currency. Furthermore, the time (or instant) can be represented as abscissas on an oriented temporal axis so as to have chronological order. The time origin is an instant fixed in a completely discretionary

---

2 In real situations, which are considered as deterministic, the stochastic component is present as a pathologic element. This component can be taken into account throughout the increase of some earning parameter or other artifices rather than introducing probabilistic elements. These elements have to be considered explicitly when uncertainty is a fundamental aspect of the problem (for example, in the theory of stochastic decision making and in actuarial mathematics). We stress that in the recent development of this subject, the aforementioned distinction, as well as the distinction between "actuarial" and "financial" mathematics, is becoming less important, given the increasing consideration of the stochastic aspect of financial problems.

way and the measure unit is usually a year (but another time measure can be used). Therefore, even the times X, Y, etc., have an algebraic sign, which is negative or positive according to their position with respect to the time origin. It follows that "$X<Y$" means "time X before time Y".

From a geometric viewpoint, we introduce in the plane $\Sigma^{(2)}$ the Cartesian orthogonal reference system OXS (with abscissas X and ordinate S). $\Sigma^{(2)}$ is then made of the points $P \equiv [X,S]$ that represent the supply (X,S), that is the amount S dated in X.

As a consequence of the postulates a) and b), the following operative criteria can be derived:

c) given two financial supplies (X,S_1) and (X,S_2) at the same maturity date X, the one with the higher (algebraically speaking) amount is preferred;

d) given two financial supplies (X,S) and (Y,S) with the same amount S and valued at instant Z before both X and Y, if $S>0$ (that is, from the cashing viewpoint) the supply for which the future maturity is closer to Z is preferred; if $S<0$ (that is, from the paying viewpoint) the supply with future maturity farther from Z is preferred. More generally $\forall Z$[3], with two supplies having the same amount, the person who cashes (who pays) prefers the supply with prior (with later) time of payment.

Formulations c) and d) express criteria of absolute preference in the financial choices and clarify the meaning of interest. In fact, referring to a loan, where the *lender* gives to the *borrower* the availability of part of his capital and its possible use for the duration of the loan, the lender would perform a disadvantageous operation (according to postulate a) and b) and criteria c) if, when the borrower gives back the borrowed capital at the fixed maturity date, he would not add a generally positive amount to the lender, which we called *interest*, as a payment for the financial service.

The decision maker's behavior is then based on preference or indifference criteria, which is subjective, in the sense that for them there is *indifference* between two supplies if neither is preferred.

To provide a better understanding of these points, we can observe that:

– the decision maker expresses a judgment of *strong preference*, indicated with $\succ$, of the supply (X_1,S_1) compared to (X_2,S_2) if he considers the first one more advantageous than the second; we then have $(X_1,S_1)\succ(X_2,S_2)$;

3 It is known that the symbol $\forall$ has the meaning "for all".

– the decision maker expresses a judgment of *weak preference,* indicated with $\succeq$, of the supply (X_1,S_1) compared to (X_2,S_2), if he does not consider the second one more advantageous than the first; we then have $(X_1,S_1)\succeq(X_2,S_2)$[4].

The amplitude of the set of supplies comparable with a given supply for a preference judgment depends on the criteria on which the judgment is based.

Criteria c) and d) make it possible to establish a preference or no preference of (X_0,S_0), but only with respect to a subset of all possible supplies, as we show below.

From a geometric point of view, let us represent the given supply (X_0,S_0) on the plane $\Sigma^{(2)}$, with reference system OXS, by the point $P_0\equiv[X_0,S_0]$. Then, considering the four quadrants adjacent to P_0, based only on criteria c) and d), it turns out that:

1) Comparing $S_0>0$ to supplies with a positive amount, identified by the points P_i (i=1,…,4) (see Figure 1.1), being incomes, it is convenient to anticipate their collection. Therefore, all points $P_2\equiv[X_2,S_2]$ in the 2nd quadrant (NW) are preferred to P_0 because they have income S_2 greater than S_0 and are available at time X_2 previous to time X_0; whereas P_0 is preferred to all points $P_4\equiv[X_4,S_4]$ in the 4th quadrant (SE) because they have income S_4 smaller than S_0 and are available at time X_4 later than X_0; it is not possible to conclude anything about the preference between P_0 and points $P_1\equiv[X_1,S_1]$ in the 1st quadrant (NE) or points $P_3\equiv[X_3,S_3]$ in the 3rd quadrant (SW).

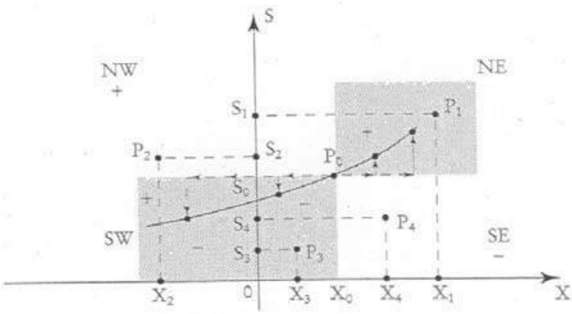

Figure 1.1. *Preferences with positive amounts*

---

4 The judgment of *weak preference* is equivalent to the merging of *strong preference* of (X_1,S_1) with respect to (X_2,S_2) and of (X_2,S_2) with respect to (X_1,S_1). In other words:

– weak preference = strong preference or indifference;

– indifference = no strong preference of one supply with respect to another.

The economic logic behind the postulates a), b), from which the criteria c), d) follow, implies that the amounts for indifferent supply have the same sign (or are both zero).

2) Comparing $S_0<0$ to supplies with a negative amount, identified by the points P_i (i=1,…,4) (see Figure 1.2), being outcomes, it is convenient to postpone their time of payment. Therefore all points $P_1 \equiv [X_1,S_1]$ in the 1st quadrant (NE) are preferred to P_0 because they have outcome S smaller than S_0 and are payable at time X later than X_0; whereas P_0 is preferred to all points $P_3 \equiv [X_3,S_3]$ in the 3rd quadrant (SW) because they have outcome S_3 greater than S_0 and are payable at time X_3, which is later than X_0. Nothing can be concluded on the preference between P_0 and all points $P_2 \equiv [X_2,S_2]$ of the 2nd quadrant (NW) or all points $P_4 \equiv [X_4,S_4]$ of the 4th quadrant (SE).

Briefly, on the non-shaded area in Figures 1.1 and 1.2 it is possible to establish whether or not there is a strong preference with respect to P_0, while on the shaded area this is not possible.

To summarize, indicating the generic supply *(X,S)* also with point $P \equiv [X,S]$ in the plane *OXS*, we observe that an operator, who follows only criteria c) and d) for his valuation and comparison of financial supplies, can select some supplies *P'* with *dominance* on P_0 (we have dominance of *P'* on P_0 when the operator prefers *P'* to P_0) and other supplies *P" dominated* by P_0 (when he prefers P_0 to *P"*), but the comparability with P_0 is *incomplete* because there are infinite supplies *P'"* not comparable with P_0 based on criteria c) and d). To make the comparability of P_0 with the set of all financial supplies *complete*, corresponding to all points in the plane referred to *OXS*, it is necessary to add to criteria c) and d) – which follow from general behavior on the ownership of wealth and the earning of interest – rules which make use of subjective parameters. The search and application of such rules – to fix them external factors must be taken into account, summarized in the “market”, making it possible to decide for each supply if it is dominant on P_0, indifferent on P_0 or dominated by P_0 – is the aim of the following discussion.

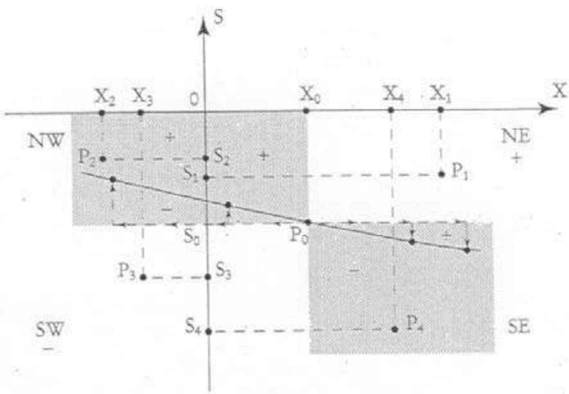

Figure 1.2. *Preferences with negative amounts*

To achieve this aim it is convenient to proceed in two phases:

1) the first phase is to select, in the zone of no dominance (shaded in Figures 1.1 and 1.2), the supplies $P^* \equiv [X^*,S^*]$ with different times of payment from that of P_0 and in indifference relation with P_0;

2) the second phase, according to the transitivity of preferences, is to select the advantageous and disadvantageous preferences with respect to P_0, with any maturity.

In the first phase, we can suppose an opinion poll on the financial operator to estimate the amount B payable in Y that the same operator evaluates in indifference relation, indicated through the symbol $\approx$, with the amount A payable in X. For such an operator we will use:

$$(X,A) \approx (Y,B) \tag{1.1}$$

Given the supply (X,A), on varying Y the curve obtained by the points that indicate the supplies (Y,B) indifferent to (X,A), or satisfying (1.1), is called the *indifference curve characterized by point* $[X,A]$.

From an operative viewpoint, if two points $P' \equiv [X,A]$ and $P'' \equiv [Y,B]$ are located on the same indifference curve, the corresponding supplies (X,A) and (Y,B) are exchangeable without adjustment by the contract parties.

If (1.1) holds, according to criteria c) and d), the amounts A and B have the same sign and $|B|-|A|$ has the same sign of Y-X. The fixation of the indifferent amounts can proceed as follows, as a consequence of the previous geometric results (see Figures 1.1 and 1.2).

Let us denote by $P_0 \equiv [X_0,S_0]$ the point representing the supply for which the indifference is searched. Then:

– if $S_0>0$ (see Figure 1.3), with $X=X_0$, $Y=X_1>X_0$, the rightward movement from P_0 to $A_1 \equiv [X_1,S_0]$ is disadvantageous because of the income delay; to remove such disadvantage the amount of the supply must be increased. The survey, using continuous increasing variations, fixes the amount $S_1>S_0$ which gives the compensation, where P_0 and $P_1 \equiv [X_1,S_1]$, obtained from A_1 moving upwards, and represents indifferent supply (or, in brief, P_1 and P_0 are indifferent points). Instead, if $Y=X_3<X_0$, the leftwards movement from P_0 to $A_3 \equiv [X_3,S_0]$ is advantageous for the income anticipation; therefore, in order to have indifference, there needs to be a decrease in the income from S_0 to S_3, obtained through a survey with downward movement of the indifference point $P_3 \equiv [X_3,S_3]$ with $S_3<S_0$;

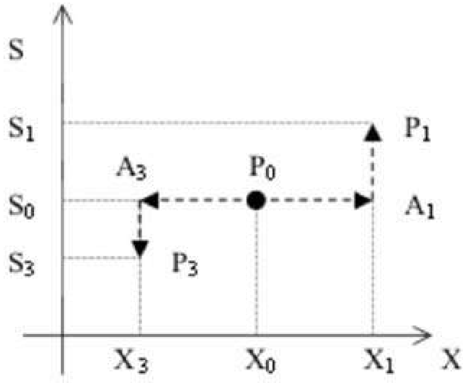

Figure 1.3. *Indifference curve assessment – positive amounts*

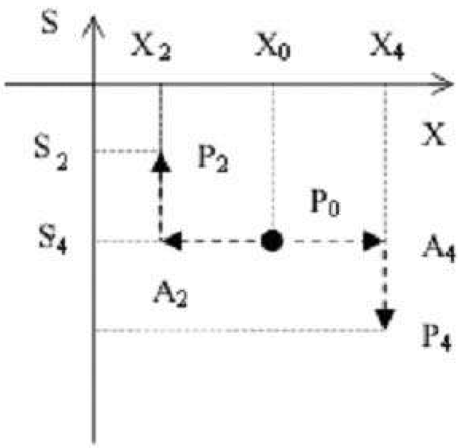

Figure 1.4. *Indifference curve assessment – negative amounts*

– if $S_0<0$ (see Figure 1.4), since the delay of outcome is advantageous and its anticipation is disadvantageous, proceeding in a similar way starting from $A_2 \equiv [X_2,S_0]$ and $A_4 \equiv [X_4,S_0]$, the points (indifferent to P_0) $P_2 \equiv [X_2,S_2]$, with $X_2<X_0$, $S_2>S_0$, are obtained through leftwards and then upwards movement, or $P_4 \equiv [X_4,S_4]$, with $X_4>X_0$, $S_4<S_0$, through rightwards and then downwards movement.

Continuously increasing or decreasing the abscissas X_i (i=1,3), we obtain, if $S_0>0$, a continuous curve with increasing ordinate in the plane *OXS*, resulting from P_0 and the points of type P_1 and P_3, all indifferent to P_0. If $S_0<0$, the continuous curve resulting by P_0 and the points of type P_4 and P_2 , all indifferent to P_0, obtained by continuously varying X_i (i=2,4), have a decreasing ordinate[5]. However, if P_0 is fixed, these curves of indifference are individualized from P_0 by definition.

We can now define, in general terms, the interest defined in section 1.1, considering only the positive amount. If (1.1) holds with $X<Y$, the exchange between indifferent supplies implies that giving away the availability of amount A from X to Y is fairly compensated by the payment of the amount

$$I = B - A \geq 0. \tag{1.2}$$

We will say that A is the invested *principal*, I is the *interest*, and B is the *accumulated value*, in an operation of *lending* or *investment*.

If (1.1) holds with $X>Y$, the anticipation of the income of A from X to Y is fairly compensated by the payment in Y of the amount

$$D = A - B \geq 0 \tag{1.3}$$

We will say that A is the *capital at maturity*, D is the *discount* and B is the *present value* or *discounted value*, in an operation of *discounting* or *anticipation*.

The second phase is applied in an easy way. It is enough to add, referring to (1.1) in the case A>0, that if a generic $P\equiv[Y,B]$ is indifferent to a fixed $P_0\equiv[X,A]$ then all the points $P'\equiv[Y,B']$ where $B'>B$ are preferred to P_0, while P_0 is preferred to all points $P''\equiv[Y,B'']$ where $B''< B$. This leads to the conclusion that, once the indifference curve through P_0 is built, all the supplies of the type (Y,B') are preferred to the supply (X,A), while the opposite occurs for all supplies of type (Y,B'').

1.2.2. *Objective aspects of financial laws. The equivalence principle*

The previous considerations enable us to give a first empirical formulation of the fundamental "principle of financial equivalence", which is that it is *equivalent*[6] *to a*

---

5 If criteria d is removed, supplies with same amount and different time become indifferent and the indifference curves have constant ordinate. All loans without interest made for free are contained in this category.

6 "Equivalent" is often used instead of "indifferent"; if this does not make sense then imagine that *P' equivalent to P"* means that these supplies are in the same equivalence class as in the set theory meaning. For this to be true, other conditions are needed. which we will discuss later.

cash (pay) amount today or to cash (pay) at a later time if there is the cashing (payment) of the interest for such deferment.

In Chapter 2, the indifference curves and the principle of financial indifference will be formalized in objective terms, defining financial factors, rates and intensities for lending and discounting operations, in relation to the possible distribution of interest payments in the deferment period. The equivalence principle will then become objective, assuming the hypothesis that different parties to a financial contract agree in fixing a rule, valid for them, to calculate the equivalent amount B, according to the amount A and the times X, Y.

1.3. The dimensional viewpoint of financial quantities

In financial mathematics, as in physics, it is necessary to introduce, together with numerical measures, a dimensional viewpoint distinguishing between *fundamental quantities* and *derived quantities.*

To describe the laws of mechanics, the oldest of the physical sciences, the following *fundamental quantities* are introduced: length l, time t, mass m, with their units (meter, second, mass-kilogram) and the *derived quantities* are deduced, such as volume l^3, velocity l/t, acceleration l/t^2, force ml/t^2, etc. Their units are derived from those of the fundamental quantity. We then speak about the physical dimension of different quantities, which are completely defined when they are given the dimensions and the numbers which represent the measurement of the given quantity in the unit system.

In financial mathematics we also make a distinction between *fundamental quantities* and *derived quantities.*

The *fundamental quantities* are:

1) *monetary* amount (m), to measure the value of financial transaction in a given unit (i.e., dollar, euro, etc.);

2) *time* (t), to measure the length of the operation and the delay of its maturity in a given unit (i.e. year).

The *derived quantities,* relating to the fundamental quantities based on dimensions, are:

1) *flow*, defined as amount over time (then with dimension m^1t^{-1});

2) *rate*, defined as amount over amount (thus a “pure number”, with dimension m^0t^0);

3) *intensity*, defined as amount over the product of amount multiplied by time (then with dimension m^0t^{-1}).

To clarify:

– *flow* relates the monetary amount to the time interval in which it is produced; a typical flow is the monetary income (i.e.: wages, fees, etc.) expressed as the monetary amount matured in a unit of time as a consequence of the considered operation;

– *rate* relates two amounts which are connected and thus is a “pure number” without dimension; for example, the rate is the ratio between matured interest and invested principal;

– *intensity*, obtained as the ratio between rate and time or flow and amount, takes into account the time needed for the formation of an amount due to another amount; for example, the ratio between interest and invested principal time length of the investment.

This is all summarized in the following *dimensional table* where we go from left to right, dividing by a “time” and from top to bottom, dividing by an “amount”.

amount (m^1t^0)	flow (m^1t^{-1})
rate (m^0t^0)	intensity (m^0t^{-1})

Table 1.1. *Financial dimensions*